Calculus AB **Chapter 6.2 Notes: Solving Differential Equation Word Problems**

1. Direct proportion equation :

2. Inverse (indirect) proportion equation:

3. k is called the ______________________________

Exponential Growth/Decay class examples

1) If the rate of change of y varies directly with the value of y, find the general equation:

2) The rate of increase of the population of a certain city is proportional to the population. If the population in 1930 was 50,000 and in 1960 it was 75,000, what was the expected population in 1990?

3) The rate of decay of radium is proportional to the amount present at any time. If 60 mg of radium are present now and its half-life is 1690 years, how much radium will be present 100 years from now?

4. In a certain culture where the rate of growth of bacteria is proportional to the amount present, the number triples in 3 hours.

A) If at the end of 12 hours there were 10 million bacteria, how many were present initially?

B) Find the specific exponential growth equation

<u>Note for homework</u>: Newton's law of cooling: the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium

6.2 Notes Differential Equations Word Problems (Exponential Growth/Decay)

1) Direct proportion equation: $y = kx$

2) Inverse (Indirect) proportion equation: $y = \frac{k}{x}$

* k is called the constant of proportionality

Ex. 1 If the rate of change of y varies directly with the value of y, find the general equation.

$y' = ky$

$\frac{dy}{dt} = ky$

$\int \frac{dy}{y} = \int k\,dt$

$\ln|y| = kt + C$

$e^{\ln|y|} = e^{kt+C}$

$y = e^{kt} \cdot e^{C}$

$y = e^{kt} \cdot C$

$\boxed{y = Ce^{kt}}$

This form will come up frequently. You may be familiar with $A = Pe^{rt}$ compounding interest formula

Ex. 2 The rate of increase of the population of a certain city is proportional to the population. If the population in 1930 was 50,000 and in 1960 it was 75,000, find the expected population in 1990.

Let $t=0$ represent 1930. Therefore $t=30$ for 1960 and $t=60$ for 1990

$P' = kP$

$\frac{dP}{dt} = kP$

$\int \frac{dP}{P} = \int k\,dt$

$\ln|P| = kt + C$

$e^{\ln|P|} = e^{kt+C}$

$P = e^{kt} \cdot e^{C}$

$P = e^{kt} \cdot C$

$P = Ce^{kt}$

*Use given information to solve for C, then k.

given ordered pairs representing (year, population)
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad t \quad , \quad P$

$(0, 50{,}000)$

$(30, 75{,}000)$

$(60, \_\_\_\_)$

$50{,}000 = Ce^{k(0)}$ $\quad$ $50{,}000 = C(1)$

$P = 50{,}000e^{kt}$ ← Now solve for k using $(30, 75{,}000)$

$75{,}000 = 50{,}000e^{k(30)}$

$1.5 = e^{30k}$

$\ln(1.5) = \ln e^{30k}$

$\ln(1.5) = 30k$

$\frac{\ln(1.5)}{30} = k$

$P = 50{,}000e^{\frac{1}{30}(\ln 1.5)t}$

*plug in $t=60$ to find population.

$P = 50{,}000e^{1/30(\ln 1.5)(60)}$

$\boxed{P = 112{,}500}$

6.2 Notes (continued)

$\boxed{\text{Ex. 3}}$ The rate of decay of radium is proportional to the amount present at any time. If 60 mg of radium are present now and its half-life is 1690 years, how much radium will be present 100 years from now?

($\overset{t}{\text{year}}$, $\overset{r}{\text{radium amount}}$)

(0, 60)

(100, ___)

(1690, 30)

$r' = kr$

$\frac{dr}{dt} = kr$

$\int \frac{dr}{r} = \int k\,dt$

$\ln|r| = kt + C$

$e^{\ln|r|} = e^{kt + C}$

$r = e^{kt} \cdot e^{c}$

$r = Ce^{kt}$

$r = Ce^{kt}$

* solve for C using (0, 60)

$60 = Ce^{k(0)}$

$\underline{60 = C}$

$r = 60e^{kt}$

* solve for k using (1690, 30)

$30 = 60e^{k(1690)}$

$\frac{30}{60} = e^{1690k}$

$0.5 = e^{1690k}$

$\ln(0.5) = \ln e^{1690k}$

$\ln(0.5) = 1690k$

$\underline{\frac{\ln 0.5}{1690} = k}$

$r = 60e^{\frac{\ln 0.5}{1690}t}$

* plug in t = 100 and solve for r

$r = 60e^{\frac{\ln 0.5}{1690}(100)}$

$\boxed{r = 57.589 \text{ mg}}$

b'

Example 4: In a certain culture where the rate of growth of bacteria is proportional to the amount present, the number triples in 3 hours. If at the end of 12 hours there were 10 million bacteria, how many were present initially? write specific equation:

$(\overset{t}{\text{time}}, \overset{b}{\text{bacteria}})$

$(0, C)$

$(3, 3C)$

$(12, 10)$

$b' = kb$

$\frac{db}{dt} = kb$

$\frac{db}{b} = k\,dt$

$\int \frac{db}{b} = \int k\,dt$

$\ln|b| = kt + c$

$e^{\ln|b|} = e^{kt + c}$

$|b| = e^{kt} \cdot e^{c}$

$b = e^{kt} \cdot C$

$b = Ce^{kt}$

* Solve for k first

$b = Ce^{kt}$

$3\cancel{C} = \cancel{C}e^{k(3)}$

$3 = e^{3k}$

$\ln 3 = \ln e^{3k}$

$\ln 3 = 3k \cancel{\ln e}$

$\frac{\ln 3}{3} = k$

$b = Ce^{(\frac{1}{3}\ln 3)t}$

* Now solve for C using (12, 10)

$10 = Ce^{(\frac{1}{3}\ln 3)(12)}$

$10 = Ce^{4\ln 3}$

$\frac{10}{e^{4\ln 3}} = C$

$\boxed{C \approx 0.123 \text{ million bacteria or } 123{,}000 \text{ bacteria}}$

b) since $C = \frac{10}{e^{4\ln 3}} = 10e^{-4\ln 3}$

$b = 10e^{-4\ln 3} \cdot e^{\frac{1}{3}\ln 3 t}$

$\boxed{b = 10e^{-4\ln 3 + (\frac{1}{3}\ln 3)t}}$

6.2 Homework p. 418-419 #17-25 odd, 33, 35, 57, 59, ~~63~~, 71, 73-76 all

Differential Equation Word Problems
(Exponential Growth/Decay)

21) Rate of change of y is proportional to y. When $x=0, y=4$ and when $x=3, y=10$. Find the value of y when $x=6$.

$y' = ky$

$\frac{dy}{dx} = ky$

$\frac{dy}{y} = k\,dx$

$(0,4)$

$(3,10)$

$(6, \_\_)$

$\int \frac{dy}{y} = \int k\,dx$

$\ln|y| = kx + C$

$e \qquad e$

$y = e^{kx} \cdot e^{c}$

$y = Ce^{kx}$

$4 = Ce^{k(0)}$

$4 = C$

$y = 4e^{kx}$

$10 = 4e^{k(3)}$

$\frac{10}{4} = e^{3k}$

$2.5 = e^{3k}$

$\ln 2.5 = \ln e^{3k}$

$\ln 2.5 = 3k$

$\frac{\ln 2.5}{3} = k$

$y = 4e^{\frac{\ln 2.5}{3}x}$

*plug $x=6$ to solve y

$y = 4e^{\frac{\ln 2.5}{3}(6)}$

$\boxed{y = 25}$

23) Rate of change of V is proportional to V. When $t=0, V=20{,}000$ and when $t=4$, $V=12{,}500$. Find value of V when $t=6$.

(t, V)

$(0, 20{,}000)$

$(4, 12{,}500)$

$(6, \_\_)$

$V' = kV$

$\frac{dV}{dt} = kV$

$\int \frac{dV}{V} = \int k\,dt$

$\ln|V| = kt + C$

$e \qquad e$

$V = e^{kt} \cdot e^{c}$

$V = Ce^{kt}$

$20{,}000 = Ce^{0}$

$V = 20{,}000e^{kt}$

$12{,}500 = 20{,}000e^{4k}$

$0.625 = e^{4k}$

$\ln 0.625 = \ln e^{4k}$

$\ln 0.625 = 4k$

$\frac{\ln 0.625}{4} = k$

$V = 20000e^{\frac{\ln 0.625}{4}t}$

*plug in $t=6$ and solve for V

$V = 20{,}000e^{\frac{\ln 0.625}{4}(6)}$

$\boxed{V = 9882.118}$

6.2 Homework (continued)

35) Radium has a half-life of 1,599 yrs. Given that after 10,000 yrs only 0.5g remain, Find the @ initial quantity and ⓑ amount after 1000 yrs.

$(0, C)$ initial amount

$(1599, \frac{1}{2}C)$

$y = Ce^{kt}$ solve for k first

$\frac{1}{2}\cancel{C} = \cancel{C}e^{1599k}$

$\frac{1}{2} = e^{1599k}$

$\ln 0.5 = \ln e^{1599k}$

$\ln 0.5 = 1599k$

$\frac{\ln 0.5}{1599} = k$

$(10{,}000, 0.5)$

$y = Ce^{\frac{\ln 0.5}{1599}t}$ plug in to solve for C

$0.5 = Ce^{\frac{\ln 0.5}{1599}(10{,}000)}$

$0.5 = C(0.13103)$

$\frac{0.5}{0.13103} = C$

$C = 38.158$

$y = 38.158e^{\frac{\ln 0.5}{1599}t}$

$y = 38.158e^{\frac{\ln 0.5}{1599}t}$

*plug in $t = 1000$ to solve for y

$y = 38.158e^{\frac{\ln 0.5}{1599}(1000)}$

$\boxed{y = 24.736\,g}$

57) Find growth model for Bulgaria and let $t=0$ represent year 2000 (time in yrs t, population in millions P)

$(1, 7.7)$

$K = -0.009$

$P = Ce^{kt}$

$7.7 = Ce^{-0.009(1)}$

$7.7 = Ce^{-0.009(1)}$

$\frac{7.7}{e^{-0.009}} = C$

$C = 7.7696$

$\boxed{a) P = 7.7696e^{-0.009t}}$

b) Use model to predict population in 2015 plug in $t=15$, solve for P.

$P = 7.7696e^{-0.009(15)}$

$\boxed{P = 6.788 \text{ million}}$

c) Since $K < 0$, the population is decreasing.

6.2 Homework (continued)

71) <u>Newton's law of cooling</u>: the rate of change in temperature of an object is proportional to the difference between the object's temperature and temperature of the surrounding medium.

When object is removed from furnace and placed in constant 80°F, medium temp. the core temperature is 1500°F. One hour later the core temperature is 1120°F. Find the core temperature 5 hrs. later.

(time, temperature)

(t, y)

$(0, 1500)$

$(1, 1120)$

$(5, \_\_)$

$$\frac{dy}{dt} = k(y-80)$$

$$\int \frac{dy}{y-80} = \int k\,dt$$

$u = y-80$

$\frac{du}{dy} = 1 \quad du = dy$

$$\int \frac{du}{u} = \int k\,dt$$

$$\ln|y-80| = kt + c$$

$e \qquad e$

$$y-80 = e^{kt} \cdot e^{c}$$

$$y-80 = e^{kt} \cdot C$$

$$y-80 = Ce^{kt}$$

$$1500-80 = Ce^{k(0)}$$

$$\underline{1420 = C}$$

$$y-80 = 1420e^{kt}$$

$$1120-80 = 1420e^{k(1)}$$

$$1040 = 1420e^{k}$$

$$0.7324 = e^{k}$$

$$\ln 0.7324 = \ln e^{k}$$

$$\ln 0.7324 = k$$

$$y-80 = 1420e^{\ln 0.7324 t}$$

*plug in $t=5$ and solve for y.

$$y-80 = 1420e^{\ln 0.7324(5)}$$

$$y-80 = 299.247$$

$$\boxed{y = 379.247^{\circ}}$$

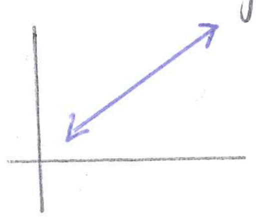
6.2 Homework (continued)

Determine if statement is True or False.

73) In exponential growth, rate is constant → $\boxed{\text{FALSE}}$

$y = Ce^{kt}$ $\quad y' = C \cdot e^{kt} \cdot k$ is not a constant with variable existing in derivative function.

74) In linear growth, rate of growth is constant → $\boxed{\text{TRUE}}$.



75) If prices are rising at rate of 0.5%/month, then they are rising at 6%/yr. → $\boxed{\text{TRUE}}$

· Use unit conversion: $\frac{0.5\%}{\cancel{\text{month}}} \cdot \frac{12\ \cancel{\text{month}}}{\text{year}} = 6\%/\text{year}$

76) The differential equation modeling exponential growth is $\frac{dy}{dx} = ky$ where k is constant → $\boxed{\text{TRUE}}$

↳ $\int \frac{dy}{y} = \int k\,dx \quad \ln|y| = kx + c \quad \boxed{y = Ce^{kx}}$

e e